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An Accurate Approach for Solving a Class of Fuzzy Optimal Control Problems with Nonlinear Constraints

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Abstract

Fuzzy optimal control problems have been applied in many fields such as electrical engineering, medicine, economics and environments. This fact is because that many of systems are not-crisp form in their modeling equations. In this paper, a class of fuzzy optimal control problems containing a nonlinear system and linear objective function is studied and a more accurate procedure is proposed. $\boldsymbol{\alpha}-\mathbf{cuts}$ and a new piecewise linearization approach are used to solve the optimization problem. At first, the fuzzy nonlinear optimal control problem is obtained and converted to a crisp nonlinear problem by applying defined $\boldsymbol{\alpha}-\mathbf{cut}$ sets of fuzzy system. Then, the problem is converted to an equivalent linear system using piecewise linearization method. Finally, the accuracy and effectiveness of the proposed approach is confirmed by several numerical examples.

Keywords: Nonlinear Optimal control, $\boldsymbol{\alpha}-\mathbf{cuts}$, fuzzy control, Piecewise linearization.

I. Introduction

Many of applied problems in real world are in fuzzy form. Several methods have been proposed in order to approximately solve nonlinear control systems including collocation method, measure theory, neural networks method and so on. [2] presents an efficient Newton-type scheme for the approximate on-line solution of optimization problems in optimal feedback control. Nonlinear model predictive control (NMPC) is studied in [3]. Gerdts et.al have developed a direct multiple shooting method in [4] for a broad and important class of DAE optimal control problems, i.e. semi explicit systems with algebraic variables of different index. Also, researchers have used collocation method for some class of fuzzy optimal control problems [5, 6] and others have used measure-theoretical approaches [7-10] and discretization methods [11, 12] for solving fuzzy optimal control problems. [13-16] discuss on using numerical methods and approximation theory techniques and neural networks methods to obtain approximate numerical solution of some classes of fuzzy optimal control problems [17-19].

Fuzzy control is a practical alternative for a variety of challenging control applications because it provides a convenient method for constructing nonlinear controllers via the use of heuristic information. Fuzzy control systems are developed based on fuzzy mathematics. Consideration of uncertainties is inevitable in many practical systems, thus, it is very

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important to develop control methods for systems having uncertainties.

Recently, control of nonlinear systems has been continuously progressed due to the demands of practical applications. Defining fuzzy logics resulted in new approaches of nonlinear control algorithms where linguistic rules, expert knowledge determine the control strategies [1]. Also, great research studies have been published on these issues [20-26]. Pakdaman and Effati have worked on solving optimal control problems using fuzzy systems in [27]. To solve an optimal control problem, the well-known Euler–Lagrange conditions are obtained and then, the solution of these conditions is approximated by defining a trial solution based on fuzzy systems. The parameters of fuzzy systems are adjusted by an optimization algorithm. Many successful applications of fuzzy control in industrial processes have been reported [28-29].

By considering a fuzzy optimal control problem as (1),

$$Min \int_{t_0}^{t_f} \varphi(f(\tilde{x}(t),\tilde{u}(t),t))dt \quad (1)$$
$$s.t$$
$$\dot{\tilde{x}} \cong g(\tilde{x}(t),u(t),t)$$
$$\tilde{x}(t_0)=\tilde{x}_0 \,, \tilde{x}(t_f)=\tilde{x}_f$$

where, $\tilde{x}\in\mathcal{F}(\mathbb{R}^n)$, $t\in[t_0,t_f]$ and $u\in U\subseteq\mathcal{F}(\mathbb{R}^m)$ that U is a compact subset of $\mathcal{F}(\mathbb{R}^m)$. Also, $\varphi:\mathcal{F}(\mathbb{R}^n)\times U\times\mathbb{R}\to\mathbb{R}$ and $g:\mathcal{F}(\mathbb{R}^n)\times U\times\mathbb{R}\to\mathcal{F}(\mathbb{R}^n)$.

optimum conditions, known as ponteryagin's minimum principle, for fuzzy optimal control problems have been derived in [36] based on the concept of differentiability and integrability of a fuzzy mapping. [37] investigates a fuzzy adaptive optimal output feedback control problem for nonlinear continuous time interconnected systems. [38] is devoted to investigation of minimum weight vertex-covering problem with uncertain vertex weights. Also, [39] analyzes the vertex coloring problem in an uncertain graph in which all vertices are deterministic, while all edges are not deterministic and there are some degree of belief in uncertain measures.

A class of fuzzy optimal control problem with crisp control function (u(t)), and fuzzy constraints can be written as (2),

$$Min \int_{t_0}^{t_f} C(t)u(t)dt$$
$$s.t$$
$$\dot{\tilde{x}} = g(\tilde{x}(t),u(t),t)$$
$$\tilde{x}(x_0)=\tilde{x}_0 \,,\ \tilde{x}\left(t_f\right)=\tilde{x}_f \quad (2)$$

Where, $\tilde{x}\in\mathcal{F}(\mathbb{R}^n)$, $t\in[t_0,t_f]$ and $u\in U\subseteq\mathbb{R}^m$ that U is a compact subset of $\mathbb{R}^m$. Also, $g(\cdot)$ is a smooth or non-smooth function on $\mathcal{F}(\mathbb{R}^n)\times U\times[t_0,t_f]$.

Moreover, there exists a pair of state and control variables $(\tilde{x}(\cdot),u(\cdot))$ such that satisfies (2) and boundary conditions $\tilde{x}(x_0)=\tilde{x}_0$, $\tilde{x}(t_f)=\tilde{x}_f$.

At first, the considered fuzzy nonlinear optimal control problem is obtained and converted to a crisp nonlinear problem by applying defined $\alpha-$cut sets of fuzzy system. Then, the problem is converted to an equivalent linear system using piecewise linearization method. Finally, the accuracy and effectiveness of the proposed approach is confirmed by several numerical examples.

II. PRELIMINARIES

According to Zadeh (1965) [1], a fuzzy set is a generalization of a classical set that allows membership function to take any value in the unit interval $[0,1]$. The formal definition of a fuzzy set is as follow:

Definition 2.1.[40] Let U be a universal set. A fuzzy set $\widetilde{A}$ in U is defined by a membership function $\mu_{\widetilde{A}}$ that maps every element in U to the unit interval [0,1]. A fuzzy set $\widetilde{A}$ in U may also be presented as a set of ordered pairs of a generic element x and its membership value, as shown in the following equation:

$$\widetilde{A}=\left\{\left(x,\mu_{\widetilde{A}}(x)\right)\ :\ x\in U\right\} \quad (3)$$

Definition 2.2.[40] The (crisp) set of elements that belong to a fuzzy set $\widetilde{A}$ at least to the degree α is called the $\alpha-$level set:

$$A_\alpha = \{x \in U \mid \mu_{\tilde{A}}(x) \geq \alpha\} \quad (4)$$

Definition 2.3.[40] A fuzzy set $\tilde{A}$ is convex if

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\} \;;\; x_1, x_2 \in U, \lambda \in [0,1] \quad (5)$$

Alternatively, a fuzzy set is convex if all α-cut sets are convex.

Definition 2.4.[40] A fuzzy number $\tilde{A}$ is a convex normalized set $\tilde{A} = (\underline{A}, \overline{A})$ of the real line $\mathbb{R}$ such that:

1) It exist exactly one $x_0 \in \mathbb{R}$ with $\mu_{\tilde{A}}(x) = 1$.

2) $\mu_{\tilde{A}}(x)$ is piecewise continuous.

Nowadays, several modifications have been proposed for definition 2.4.

Definition 2.5.[41] Fuzzy set $\tilde{A}$ is called a fuzzy number if :

1. $\tilde{A}$ is normal.
2. $\tilde{A}$ is convex.
3. $\mu_{\tilde{A}}(x)$ is piece wise continuous.

Thorem 2.6.[41] A fuzzy number $\tilde{A}$ is a Ordered pair $[A]^\alpha = [\underline{A}^\alpha, \overline{A}^\alpha]$ of function $\underline{A}(\alpha), \overline{A}(\alpha): [0,1] \to \mathbb{R}$ that $0 \leq \alpha \leq 1$ and satisfies below conditions:

1. $\underline{A}(\alpha)$ is a nondecreasing and left-continuous function on $[0,1]$.
2. $\overline{A}(\alpha)$ is a nonincreasing and right-continuous function on $[0,1]$.
3. $\underline{A}(\alpha) \leq \overline{A}(\alpha)$ for any $0 \leq \alpha \leq 1$.

Definition 2.7.[41] Let $L(.), R(.): \mathbb{R}^+ \to [0,1]$ be two continuous, increasing functions fulfilling

$$L(0) = R(0) = 1, L(1) = R(1) = 0$$

A fuzzy number $\tilde{A}$ is a $L-R$ fuzzy number, if there exist $\alpha, \beta > 0$ that

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right) & , \ x \leq m \\ R\left(\frac{x-m}{\beta}\right) & , \ x \geq m \end{cases} \quad (6)$$

Definition 2.8. Set of all fuzzy number on $\mathbb{R}^n$ is shown by $F(\mathbb{R}^n)$.

Theorem. (Negoita-Ralescu characterization theorem)[41]

Given a family of subsets $\{M_\alpha : \alpha \in [0,1]\}$ that satisfies conditions (i)-(iv)

(i) M_α is a non-empty closed interval for any $\alpha \in [0,1]$;

(ii) If $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, we have $M_{\alpha_2} \subseteq M_{\alpha_1}$;

(iii) For any sequence $\{\alpha_n\}$ which converges from below to $\alpha \in [0,1]$ we have $\bigcap_{n=1}^{\infty} M_{\alpha_n} = M_\alpha$;

(iv) For any sequence α_n which converges from above to 0 we have

$$cl\left(\bigcup_{n=1}^{\infty} M_{\alpha_n}\right) = M_0 \quad (7)$$

Then there exists a unique $u \in F(\mathbb{R})$, such that $u_\alpha = M_\alpha$, for any $\alpha \in [0,1]$.

Definition 2.9.[40] A triangular fuzzy number has the following form:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{m-a} & a \leq x \leq m \\ \frac{b-x}{b-m} & m \leq x \leq b \\ 0 & b < x \end{cases} \quad (8)$$

A triangular fuzzy number is denoted by $\tilde{A} = (m, a, b)$ where, $c \neq a$, $c \neq b$. For a triangular fuzzy number, we have:

$\underline{A}(\alpha) = a + (c-a)\alpha$

and

$\overline{A}(\alpha) = b + (c-b)\alpha$

a and b are called left and right spreads of the fuzzy number $\tilde{A}$. A triangular fuzzy number is shown in Figure 1.

Definition 2.10.[40] A fuzzy number $\tilde{A}$ is called positive (negative) if its membership function is such that $\mu_{\tilde{A}}(x) = 0$, $\forall x < 0$ $(\forall x > 0)$.

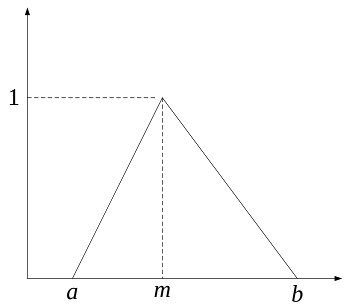

Fig. 1. The figure of triangular fuzzy number $\tilde{A}$

One of the most basic concepts of fuzzy set theory that can be used to generalize crisp mathematical concepts to fuzzy sets is the extension principle.

Definition 2.11.[40] Let $X = X_1 \times X_2 \times \dots \times X_r$ and $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_r$ be r fuzzy sets in $X_1, X_2, \dots, X_r$, respectively. f is a mapping from X to a universe Y, $y = f(x_1, x_2, \dots, x_r)$. Then the extension principle allows us to define a fuzzy set $\widetilde{B}$ in Y by

$$\widetilde{B} = \{(y, \mu_{\widetilde{B}}(y)) \mid y = f(x_1, x_2, \dots, x_r)\ ,\ (x_1, x_2, \dots, x_r) \in X\} \quad (9)$$

Where

$$\mu_{\widetilde{B}}(y) = \left\{ \sup_{(x_1,x_2,\dots,x_r)\in f^{-1}(y)} \min \{\mu_{\widetilde{A}_1}(x_1), \mu_{\widetilde{A}_2}(x_2), \dots, \mu_{\widetilde{A}_r}(x_r)\} \ \text{if } f^{-1}(y) \neq \emptyset \right\} \quad (10)$$

Where f^{-1} is the inverse of f.

For $r = 1$, the extension principle, of course, reduces to

$$\widetilde{B} = f(\widetilde{A}) = \{(y, \mu_{\widetilde{B}}(y)) \mid y = f(x)\ , x \in X\} \quad (11)$$

Where

$$\mu_{\widetilde{B}}(y) = \begin{cases} \min_{x\in f^{-1}(y)} \mu_{\widetilde{A}}(x) & , f^{-1}(y) \neq \emptyset \\ 0 & , \text{otherwise} \end{cases} \quad (12)$$

According to [40], extended addition, product and subtraction are shown by $\oplus$, $\odot$ and $\ominus$, respectively.

Theorem 2.12. [30] Let $\widetilde{A}, \widetilde{B} \in \mathcal{F}(\mathbb{R})$, then in terms of $\alpha-$cuts we have:

$\forall \alpha \in [0,1]$, we have

$$[\widetilde{A} \oplus \widetilde{B}]^\alpha = [\widetilde{A}]^\alpha + [\widetilde{B}]^\alpha = [\underline{A}^\alpha + \underline{B}^\alpha, \overline{A}^\alpha + \overline{B}^\alpha]$$

$$k[\widetilde{A}]^\alpha = \begin{cases} [k\underline{A}^\alpha, k\overline{A}^\alpha] & , k \geq 0 \\ [k\overline{A}^\alpha, k\underline{A}^\alpha] & , k < 0 \end{cases} \quad (13)$$

$$[\widetilde{A} \ominus \widetilde{B}]^\alpha = [\widetilde{A}]^\alpha \oplus (-1) \odot [\widetilde{B}]^\alpha$$

$$[\widetilde{A} \odot \widetilde{B}]^\alpha = [\min\{\underline{A}^\alpha \underline{B}^\alpha, \underline{A}^\alpha \overline{B}^\alpha, \overline{A}^\alpha \underline{B}^\alpha, \overline{A}^\alpha \overline{B}^\alpha\}, \max\{\underline{A}^\alpha \underline{B}^\alpha, \underline{A}^\alpha \overline{B}^\alpha, \overline{A}^\alpha \underline{B}^\alpha, \overline{A}^\alpha \overline{B}^\alpha\}]$$

Definition 2.13.[40] The metric structure is given by the Hausdroff distance

$D: \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathbb{R}^+ \cup \{0\}$,

$$D(u, v) = \sup_{\alpha\in[0,1]} \max\{ |\underline{u}^\alpha - \underline{v}^\alpha|, |\overline{u}^\alpha - \overline{v}^\alpha| \quad (14)$$

(R_F, D) is a complete metric space and the following properties are well known:

$$D(u + w, v + w) = D(u, v), \forall\, u, v, w \in \mathcal{F}(\mathbb{R})$$

$$D(k \odot u, k \odot v) = |k| D(u, v)\ , \forall k \in R, u, v \in \mathcal{F}(\mathbb{R}) \quad (15)$$

$$D(u + v, w + e) \leq D(u, w) + D(v, e)\ , \forall\, u, v, w, e \in \mathcal{F}(\mathbb{R})$$

Definition 2.14. (Stefanini, Stefanini-Bede) [30]. Given two fuzzy numbers $u, v \in F(\mathbb{R})$, the generalized Hukuhara difference (gH-difference for such)is the fuzzy number $w \in F(\mathbb{R})$ if it exists, such that

$$\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w} \Leftrightarrow \begin{cases} (i) & u = v + w \\ & or \\ (ii) & v = u - w \end{cases} \quad (16)$$

In term of $\alpha-$cuts we have

Theorem 2.15[30]. For any $\tilde{u}, \tilde{v} \in F(\mathbb{R})$ we have

$$[\tilde{u} \ominus_{gH} \tilde{v}]^\alpha = \begin{cases} (i) & [\underline{u}^\alpha - \underline{v}^\alpha, \overline{u}^\alpha - \overline{v}^\alpha] \\ & or \\ (ii) & [\overline{u}^\alpha - \overline{v}^\alpha, \underline{u}^\alpha - \underline{v}^\alpha] \end{cases} \quad (17)$$

Definition 2.16. [30] The mapping $f: T \to F(\mathbb{R}^n)$ that $T \subseteq \mathbb{R}$ is called a fuzzy mapping. By $\alpha - cut$ sets definition it can be written as follows

$$[f(t)]^\alpha = [\underline{f}^\alpha(t), \overline{f}^\alpha(t)] \quad ;\ t \in T \ ;\ 0 \leq \alpha \leq 1 \quad (18)$$

Definition 2.17. [30] (Puri-Ralescu , Hukuhara]) A function $f: T \to F(\mathbb{R})$ is called Hukuhara differentiable if for $h > 0$ sufficiently small the H-differences $f(t_0 + h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0 - h)$ exist and if there exist an element $f'(t_0) \in F(\mathbb{R})$,such that

$$\lim_{h\to0^+} \frac{f(t_0+h)\ominus f(t_0)}{h} = \lim_{h\to0^+} \frac{f(t_0)\ominus f(t_0-h)}{h} = f'(t_0) \quad (19)$$

The fuzzy number $f'(t_0)$ is called the Hukuhara derivative of f at t_0.

Theorem 2.18. [30] Let $f: T \to F(\mathbb{R})$ be a fuzzy function, where $[\tilde{f}(t)]^\alpha = [\underline{f}^\alpha(t), \overline{f}^\alpha(t)]$

Suppose that the functions s f $\underline{f}^\alpha(t)$ and $\overline{f}^\alpha(t)$ are real-valued functions, differentiable uniformly $\alpha \in [0,1]$ .Then the function $\tilde{f}(t)$ is gH-differentiable at a fixed $t \in T$ if and only if one of the following two cases hold:

$\dot{\underline{f}}^\alpha(t)$ is increasing $\dot{\overline{f}}^\alpha(t)$ is decreasing as functions of α, and

$$\dot{\underline{f}}^\alpha(t) \leq \dot{\overline{f}}^\alpha(t) \quad (20)$$

$\dot{\underline{f}}^\alpha(t)$ is decreasing $\dot{\overline{f}}^\alpha(t)$ is increasing as functions of α, and

$$\dot{\overline{f}}^\alpha(t) \leq \dot{\underline{f}}^\alpha(t) \quad (21)$$

Also, $\forall\ \alpha \in [0,1]$, we have

$$[\dot{f}_{gH}(t)]^\alpha = \left[\min\left\{\dot{\underline{f}}^\alpha(t), \dot{\overline{f}}^\alpha(t)\right\}, \max\left\{\dot{\underline{f}}^\alpha(t), \dot{\overline{f}}^\alpha(t)\right\}\right] \quad (22)$$

Let us denote $\|.\|_F$ the norm of fuzzy number. The follow definition shows it is not a norm but is a semi norm.

Definition 2.19.[41] (Anastassiou-Gal) $\|.\|_F$ has the following properties

(i) $\|u\|_F = 0$ if and only if $u = 0$;

(ii) $\|\lambda u\|_F = |\lambda| \|u\|_F \quad , \lambda \in \mathbb{R}, u \in F(\mathbb{R})$;

(iii) $\|u + v\|_F \leq \|u\|_F + \|v\|_F \quad , u, v \in F(\mathbb{R})$

(23)

Theorem 2.20. Let to consider $[\tilde{u}]^\alpha = (\underline{u}^\alpha, \overline{u}^\alpha)$. The mapping $\|.\|_1$ that is defined as below, is a semi norm.

$$\|[\tilde{u}]^\alpha\|_1 = \|(\underline{u}^\alpha, \overline{u}^\alpha)\|_1 = |\underline{u}^\alpha| + |\overline{u}^\alpha| \quad (24)$$

Proof.

For any $\tilde{u} \in F(\mathbb{R})$

$$\|[\tilde{u}]^\alpha\|_1 = \left\|\left(\underline{u}^\alpha, \overline{u}^\alpha\right)\right\|_1 = \left|\underline{u}^\alpha\right| + \left|\overline{u}^\alpha\right| \geq 0 \quad (25)$$

Also,

$$\|[\tilde{u}]^\alpha\|_1 = 0 \rightarrow \left\|\left(\underline{u}^\alpha, \overline{u}^\alpha\right)\right\|_1 = \left|\underline{u}^\alpha\right|_1 + \left|\overline{u}^\alpha\right|_1 = 0 \rightarrow$$
$$\left|\underline{u}^\alpha\right|_1 = 0, \left|\overline{u}^\alpha\right|_1 = 0 \quad (26)$$
$$\rightarrow \underline{u}^\alpha = 0, \overline{u}^\alpha = 0 \rightarrow [\tilde{u}]^\alpha = 0$$

For any $\lambda \in \mathbb{R}, u \in F(\mathbb{R})$

$$\|\lambda[\tilde{u}]^\alpha\|_1 = \left\|\lambda\left(\underline{u}^\alpha, \overline{u}^\alpha\right)\right\|_1 = \left\|\left(\lambda\underline{u}^\alpha, \lambda\overline{u}^\alpha\right)\right\|_1 =$$
$$\left|\lambda\underline{u}^\alpha\right|_1 + \left|\lambda\overline{u}^\alpha\right|_1$$
$$= |\lambda|\left|\underline{u}^\alpha\right|_1 + |\lambda|\left|\overline{u}^\alpha\right|_1 \quad (27)$$

Also

$$|\lambda|\|[\tilde{u}]^\alpha\|_1 = |\lambda|\left\|\left(\underline{u}^\alpha, \overline{u}^\alpha\right)\right\|_1 = |\lambda|\left(\left|\underline{u}^\alpha\right|_1 + \left|\overline{u}^\alpha\right|_1\right) =$$
$$|\lambda|\left|\underline{u}^\alpha\right|_1 + |\lambda|\left|\overline{u}^\alpha\right|_1 \quad (28)$$

Hence, relation (27) and (28) conclude

$$\|\lambda[\tilde{u}]^\alpha\|_1 = |\lambda|\|[\tilde{u}]^\alpha\|_1$$

Let $u, v \in F(\mathbb{R})$. So

$$\|[\tilde{u}]^\alpha + [\tilde{v}]^\alpha\|_1 = \left\|\left(\underline{u}^\alpha, \overline{u}^\alpha\right) + \left(\underline{v}^\alpha, \overline{v}^\alpha\right)\right\|_1$$
$$= \left\|\left(\underline{u}^\alpha + \underline{v}^\alpha, \overline{u}^\alpha + \overline{v}^\alpha\right)\right\|_1$$
$$\left|\underline{u}^\alpha + \underline{v}^\alpha\right| + \left|\overline{u}^\alpha + \overline{v}^\alpha\right| \leq \left|\underline{u}^\alpha\right| + \left|\underline{v}^\alpha\right| + \left|\overline{u}^\alpha\right| + \left|\overline{v}^\alpha\right| \quad (29)$$

Also,

$$\|[\tilde{u}]^\alpha\|_1 + \|[\tilde{v}]^\alpha\|_1 = \left\|\left(\underline{u}^\alpha, \overline{u}^\alpha\right)\right\|_1 + \left\|\left(\underline{v}^\alpha, \overline{v}^\alpha\right)\right\|_1 =$$
$$\left|\underline{u}^\alpha\right| + \left|\overline{u}^\alpha\right| + \left|\underline{v}^\alpha\right| + \left|\overline{v}^\alpha\right| \quad (30)$$

Hence, relation (29) and (30) conclude

$$\|[\tilde{u}]^\alpha + [\tilde{v}]^\alpha\|_1 \leq \|[\tilde{u}]^\alpha\|_1 + \|[\tilde{v}]^\alpha\|_1 \quad (31)$$

III. PIECEWISE LINEARIZATION

The linearization of nonlinear systems is an efficient tool for finding approximate solutions and treatment analysis of these systems. Different methods based on the optimization problem are presented in previously published papers, while this paper deals with major issues of using non-linear and non-smooth functions exist in many practical applications. Therefore, piecewise linearization is a more efficient tool for finding approximate solutions. Hence, the piecewise linearization approach, which is introduced in [34], is applied to solve the considered class of non-linear fuzzy optimal control problem of this paper.

Let $F: A \subseteq R^n \rightarrow R$ be a nonlinear function. Suppose that $x \in A \subseteq R^n$ and a subset is compact. It is aimed to approximate the nonlinear function F by a piecewise linear function as:

$$F_N(x) = \sum_{i=1}^{N}(a_{i0} + a_{i1}x_1 + a_{i2}x_2 + \cdots +$$
$$a_{in}x_n)\chi_{A_i}(x), a_{ij} \in R\,; i = 1,2,\ldots,N \quad (32)$$

where, A_i is i th subset in partitioning of A as $P_N = \{A_1, A_2, \ldots, A_N\}$. This partitioning has the following properties:

1) $\forall\, i,j = 1,2,\ldots,N;\ A_i \cap A_j = \emptyset,\ A_i \in R^n,\ A_i \neq \emptyset$
2) $A = \cup_{i=1}^{N} A_i$ (33)

At first, a class of classic nonlinear optimal control problem is considered as:

$$\text{Min} \int_{t_0}^{t_f} C(t)u(t)dt$$
$$\text{s.t:}\begin{cases} \dot{x} = g(x(t), u(t), t) \\ x(t_0) = x_0, x(t_f) = x_f \end{cases} \quad (34)$$

where, $x(t) \in \mathbb{R}^n$, $t \in [t_0, t_f]$ and $u(t) \in U \subseteq \mathbb{R}^m$ that U is a compact subset of $\mathbb{R}^m$. Also, $g(\cdot)$ is a nonlinear smooth or non-smooth function on $\mathbb{R}^n \times U \times [t_0, t_f]$.

Definition 3.1. The general error functional $E(\dot{x}(.), x(.), u(.))$ is defined as,

$$E(\dot{x}(.), x(.), u(.)) = \int_{t_0}^{t_f}\{|C(t)u(t)| + \|\dot{x}(t) -$$
$$g(x(t), u(t), t)\|_1\}dt \quad (35)$$

Lemma 3.1. If the function $h(t)$ is continuous on $[a, b]$ interval and $\int_a^b |h(t)|dt = 0$, then $h(t) = 0$.

Proof: Suppose that there exists any point $s \in (a, b)$ that $h(s) \neq 0$. Hence, $|h(s)| > 0$. Also, because $h(t)$ is continuous, hence, $|h(t)|$ is continuous.

Then, there exist $0 < r$ such that $\forall t \in (s - r, s + r)$, $r \leq \min\{s - a, b - s\} : |h(t)| > 0$. Hence, it can be concluded that because of contradiction of (36), $h(t) = 0$ on $[a, b]$ interval.

$$\int_a^b |h(t)|dt = \int_a^{s-r} |h(t)|dt + \int_{s-r}^{s+r} |h(t)|dt + \int_{s+r}^{b} |h(t)|dt >$$
$$\int_{s-r}^{s+r} |h(t)|dt > 0 \quad (36)$$

Theorem 3.1. The necessary and sufficient condition for classic optimal control problem of (34) to obtain solution, $(x(t), u(t))$, is:

$$E(\dot{x}(t), x(t), u(t), t) = 0$$

Proof. It is sufficient to consider function $h(t)$ in lemma 3.1 as,

$$h(t) = \{w_1|C(t)u(t)| + w_2\|\dot{x}(t) - g(x(t), u(t), t)\|_1\} \quad (37)$$

where,

$w_1 + w_2 = 1 \ , w_1, w_2 \geq 0.$

As the norm function $\|\cdot\|_1$, $x(\cdot)$ and $\dot{x}(\cdot)$ are continuous functions, hence, $h(t)$ is a continuous function and according lemma 3.1,

$$E(\dot{x}(t), x(t), u(t), t) = \int_{t_0}^{t_f} \{w_1 |C(t)u(t)| + w_2 \|\dot{x}(t) - g(x(t), u(t), t)\|_1\} dt = 0$$
$$\Rightarrow \{w_1 |C(t)u(t)| + w_2 \|\dot{x}(t) - g(x(t), u(t), t)\|_1\} = 0$$
$$\Rightarrow h(t) = 0 \tag{38}$$

where,

$$w_1 + w_2 = 1 \, , w_1, w_2 \geq 0$$

Now, a class of fuzzy nonlinear optimal control problem is considered a:

$$Min \int_{t_0}^{t_f} |C(t)u(t)| \, dt$$

$$s.t: \begin{cases} \dot{\tilde{x}}(t) \cong a \, \tilde{x}(t) + g(u(t), t) \\ \tilde{x}(t_0) = \tilde{x}_0 \, , \tilde{x}(t_f) = \tilde{x}_f \end{cases} \tag{39}$$

where, $\tilde{x}(t) \in A \subseteq \mathcal{F}(\mathbb{R}^n)$, $t \in [t_0, t_f]$ and $u(t) \in U \subseteq \mathbb{R}^m$ that A is a compact subset of $\mathcal{F}(\mathbb{R}^n)$ and U is a compact subset of $\mathbb{R}^m$. Also, $\dot{\tilde{x}}(.), \tilde{x}(.), u(.)$ are continues functions on $[t_0, t_f]$ and $g(\cdot)$ is a nonlinear smooth or non-smooth function on $U \times [t_0, t_f]$.

Definition 3.2. The error functional $E(\dot{\tilde{x}}(t), \tilde{x}(t), u(t), t)$ on $A \times U \times [t_0, t_f]$ is defined as,

$$E([\dot{\tilde{x}}(t)]^\alpha, [\tilde{x}(t)]^\alpha, u(t), t) = \int_{t_0}^{t_f} \left\{ w_1 |C(t)u(t)| + w_2 \left\| [\dot{\tilde{x}}(t)]^\alpha \ominus_{gH} \left(a \ [\tilde{x}(t)]^\alpha \oplus g(u(t), t) \right) \right\|_1 \right\} dt \tag{40}$$

$, w_1 + w_2 = 1 \, , w_1, w_2 \geq 0$

Theorem 3.2. The necessary and sufficient condition for fuzzy nonlinear optimal control problem of (39) to have solution, $(\tilde{x}(t), u(t))$, is

$$E([\dot{\tilde{x}}(t)]^\alpha, [\tilde{x}(t)]^\alpha, u(t), t) = 0 \tag{41}$$

Proof. It is sufficient to consider function $h(t)$ in lemma 3.1 as follows,

$$h(t) = \left\{ w_1 |C(t)u(t)| + w_2 \left\| [\dot{\tilde{x}}(t)]^\alpha \ominus_{gH} \left(a \ [\tilde{x}(t)]^\alpha \oplus g(u(t), t) \right) \right\|_1 \right\} \tag{42}$$

where,

$w_1 + w_2 = 1 \, , w_1, w_2 \geq 0.$

As the norm function $\|\cdot\|_1$, $\tilde{x}(\cdot)$ and $\dot{\tilde{x}}(\cdot)$ are continuous functions, hence, $h(t)$ is a continuos function and according lemma 3.1,

$$E([\dot{\tilde{x}}(t)]^\alpha, [\tilde{x}(t)]^\alpha, u(t), t) = \int_{t_0}^{t_f} \left\{ w_1 |C(t)u(t)| + w_2 \left\| [\dot{\tilde{x}}(t)]^\alpha \ominus_{gH} \left(a \ [\tilde{x}(t)]^\alpha \oplus g(u(t), t) \right) \right\|_1 \right\} dt = 0$$
$$\Rightarrow \left\{ w_1 |C(t)u(t)| + w_2 \left\| [\dot{\tilde{x}}(t)]^\alpha \ominus_{gH} \left(a \ [\tilde{x}(t)]^\alpha \oplus g(u(t), t) \right) \right\|_1 \right\} = 0$$
$$\Rightarrow h(t) = 0 \tag{43}$$

Now, the following optimization problem can be considered,

$$Min \int_{t_0}^{t_f} \left\{ |C(t)u(t)| + \left\| [\dot{\tilde{x}}(t)]^\alpha \ominus_{gH} \left(a \ [\tilde{x}(t)]^\alpha \oplus g(u(t), t) \right) \right\|_1 \right\} dt$$
$$s.t: \begin{cases} \tilde{x}(t_0) = \tilde{x}_0 \\ \tilde{x}(t_f) = \tilde{x}_f \end{cases} \tag{44}$$

Theorem 3.3. The necessary and sufficient condition for fuzzy nonlinear optimal control problem (39) to have solution $(\tilde{x}^*(t), u^*(t))$ is that the optimization problem (44) has zero optimal amount.

Proof. It is obvious according lemma 3.1 and theorem 3.2.

Hence, according theorem 3.2, the solution of fuzzy nonlinear optimal control problem (39) can be obtained by solving the optimization problem (44). If the optimization problem (44) does not have zero solution, then the best-approximated solution is achieved for fuzzy nonlinear optimal control problem (39).

$\alpha -$ cut sets are used to solve the optimization problem (39). Also, function $g(\cdot)$ is approximated using a new piecewise linearization definition. Hence, the following optimization problem is resulted.

$$\text{Min} \int_{t_0}^{t_f}\Big\{|C(t)u(t)| + \Big\|\big[\dot{\tilde{x}}(t)\big]^\alpha \ominus_{gH} \Big(a\ [\tilde{x}(t)]^\alpha \oplus \sum_{i=1}^N \{b_i u + c_i t + d_i\}\,\chi_{A_i}(u,t)\Big)\Big\|_1\Big\}\,dt$$
$$\text{s.t:} \begin{cases} \tilde{x}(t_0) = \tilde{x}_0 \\ \tilde{x}(t_f) = \tilde{x}_f \end{cases} \tag{45}$$

where

$$g(u(t),t) \approx \sum_{i=1}^N \{b_i u + c_i t + d_i\}\,\chi_{A_i} \text{ and } A_i \subseteq U \times [t_0, t_f],\ A_i \cap A_j = \emptyset,\ \bigcup_{i=1}^N A_i = A \tag{46}$$

…..

$$\text{Min} \int_{t_0}^{t_f}\Big\{|C(t)u(t)| + \Big\|\dot{\underline{x}}^\alpha(t) - \Big(a\,\underline{x}^\alpha(t) + \sum_{i=1}^N \{b_i u + c_i t + d_i\}\,\chi_{A_i}(u,t)\Big), \dot{\overline{x}}^\alpha(t) - \Big(a\,\overline{x}^\alpha(t) + \sum_{i=1}^N \{b_i u + c_i t + d_i\}\,\chi_{A_i}(u,t)\Big)\Big\|_1\Big\}\,dt$$
$$\text{s.t:} \begin{cases} \tilde{x}(t_0) = \tilde{x}_0 \\ \tilde{x}(t_f) = \tilde{x}_f \end{cases} \tag{47}$$

Now,(47) is discretized using by Riemann approximation and is converted optimization problem:

$$\text{Min} \sum_{j=1}^n \Big\{|C_j u_j| + \Big|\dot{\underline{x}}_j^{\ \alpha} - \Big(a\,\underline{x}_j^{\ \alpha} + \sum_{i=1}^N \{b_i u_j + c_i t_j + d_i\}\,\chi_{A_i}(u,t)\Big)\Big| + \Big|\dot{\overline{x}}_j^{\ \alpha} - \Big(a\,\overline{x}_j^{\ \alpha} + \sum_{i=1}^N \{b_i u_j + c_i t_j + d_i\}\,\chi_{A_i}(u,t)\Big)\Big|\Big\}$$
$$\text{s.t:} \begin{cases} [\tilde{x}(t_0)]^\alpha = \big(\underline{x}^\alpha(t_0), \overline{x}^\alpha(t_0)\big) \\ [\tilde{x}(t_f)]^\alpha = \big(\dot{\underline{x}}^\alpha(t_f), \dot{\overline{x}}^\alpha(t_f)\big) \end{cases} \tag{48}$$

where, $[x(t_j)]^\alpha = [x_j]^\alpha$, $u(t_j) = u_j$ and $C(t_j) = C_j$, $j = 0,1,\dots,N$, $t_j = t_0 + j\frac{t_f - t_0}{N}$

According to boundary condition relations of $[\tilde{x}(t_j)]^\alpha = \big(\underline{x}^\alpha(t_j), \overline{x}^\alpha(t_j)\big)$, $[\dot{\tilde{x}}(t_j)]^\alpha = \big(\dot{\underline{x}}^\alpha(t_j), \dot{\overline{x}}^\alpha(t_j)\big)$, $[\tilde{x}(t_0)]^\alpha = \big(\underline{x}^\alpha(t_0), \overline{x}^\alpha(t_0)\big)$, and $[\tilde{x}(t_f)]^\alpha = \big(\underline{x}^\alpha(t_f), \overline{x}^\alpha(t_f)\big)$.

For sufficient small h, we have

$$\dot{\underline{x}}_j^{\ \alpha} = \dot{\underline{x}}^\alpha(t_j) \approx \frac{\underline{x}^\alpha(t_j + h) - \underline{x}^\alpha(t_j)}{h},\quad \dot{\overline{x}}_j^{\ \alpha} = \dot{\underline{x}}^\alpha(t_j) \approx \frac{\overline{x}^\alpha(t_j + h) - \overline{x}^\alpha(t_j)}{h},\ j = 0,1,\dots,N-1. \tag{49}$$

The optimization problem of (48) can be written as:

$$\text{Min} \sum_{j=1}^n \Big\{|C_j u_j| + \Big|\frac{\underline{x}^\alpha(t_j + h) - \underline{x}^\alpha(t_j)}{h} - \Big(a\,\underline{x}_j^{\ \alpha} + \sum_{i=1}^N \{b_i u_j + c_i t_j + d_i\}\,\chi_{A_i}(u,t)\Big)\Big| + \Big|\frac{\overline{x}^\alpha(t_j + h) - \overline{x}^\alpha(t_j)}{h} - \Big(a\,\overline{x}_j^{\ \alpha} + \sum_{i=1}^N \{b_i u_j + c_i t_j + d_i\}\,\chi_{A_i}(u,t)\Big)\Big|\Big\}$$
$$\text{s.t:} \begin{cases} [\tilde{x}(t_0)]^\alpha = \big(\underline{x}^\alpha(t_0), \overline{x}^\alpha(t_0)\big) \\ [\tilde{x}(t_f)]^\alpha = \big(\dot{\underline{x}}^\alpha(t_f), \dot{\overline{x}}^\alpha(t_f)\big) \end{cases} \tag{50}$$

According $\underline{x}^\alpha(t_j + h) = \underline{x}_{j+1}^\alpha$, $\overline{x}^\alpha(t_j + h) = \overline{x}_{j+1}^\alpha$, $h = \frac{t_f - t_0}{N}$ .and

$r_j - s_j = C_j u_j$, $j = 1,\dots,N$.

$$v_j^\alpha - w_j^\alpha = \frac{\underline{x}^\alpha(t_j + h) - \underline{x}^\alpha(t_j)}{h} - \Big(a\,\underline{x}_j^{\ \alpha} + \sum_{i=1}^N \{b_i u_j + c_i t_j + d_i\}\,\chi_{A_i}(u,t)\Big) \tag{51}$$

$$k_j^\alpha - p_j^\alpha = \frac{\overline{x}^\alpha(t_j + h) - \overline{x}^\alpha(t_j)}{h} - \Big(a\,\overline{x}_j^{\ \alpha} + \sum_{i=1}^N \{b_i u_j + c_i t_j + d_i\}\,\chi_{A_i}(u,t)\Big)$$

The optimization problem of (50) can be written as:

$$\mathrm{Min}\ \sum_{j=1}^{n}\left\{\left|r_j - s_j\right| + \left|v_j^{\alpha} - w_j^{\alpha}\right| + \left|k_j^{\alpha} - p_j^{\alpha}\right|\right\}$$

$$\begin{cases} r_j - s_j = C_j u_j \quad ,j = 1,\dots,N \\ v_j^{\alpha} - w_j^{\alpha} = \dfrac{\underline{x}^{\alpha}(t_j + h) - \underline{x}^{\alpha}(t_j)}{h} - \left(\alpha\, \underline{x}_j{}^{\alpha} + \displaystyle\sum_{i=1}^{N}\{b_i u_j + c_i t_j + d_i\}\chi_{A_i}(u,t)\right) \\ v_j^{\alpha} - w_j^{\alpha} = \dfrac{\underline{x}^{\alpha}(t_j + h) - \underline{x}^{\alpha}(t_j)}{h} - \left(\alpha\, \overline{x}_j{}^{\alpha} + \displaystyle\sum_{i=1}^{N}\{b_i u_j + c_i t_j + d_i\}\chi_{A_i}(u,t)\right) \\ [\tilde{x}(t_0)]^{\alpha} = \left(\underline{x}^{\alpha}(t_0), \overline{x}^{\alpha}(t_0)\right) \\ [\tilde{x}(t_f)]^{\alpha} = \left(\dot{\underline{x}}^{\alpha}(t_f), \overline{x}^{\alpha}(t_f)\right) \end{cases}$$

Finally, the linearized form of (52) is

$$\mathrm{Min}\ \sum_{j=1}^{n}\left\{(r_j + s_j) + \left(v_j^{\alpha} + w_j^{\alpha}\right) + \left(k_j^{\alpha} + p_j^{\alpha}\right)\right\} \qquad (53)$$

$$\text{s.t:}\begin{cases} r_j - s_j = C_j u_j \quad ,j = 1,\dots,N \\ v_j^{\alpha} - w_j^{\alpha} = \dfrac{\underline{x}^{\alpha}(t_j + h) - \underline{x}^{\alpha}(t_j)}{h} - \left(\alpha\, \underline{x}_j{}^{\alpha} + \displaystyle\sum_{i=1}^{N}\{b_i u_j + c_i t_j + d_i\}\chi_{A_i}(u,t)\right) \\ v_j^{\alpha} - w_j^{\alpha} = \dfrac{\underline{x}^{\alpha}(t_j + h) - \underline{x}^{\alpha}(t_j)}{h} - \left(\alpha\, \overline{x}_j{}^{\alpha} + \displaystyle\sum_{i=1}^{N}\{b_i u_j + c_i t_j + d_i\}\chi_{A_i}(u,t)\right) \\ [\tilde{x}(t_0)]^{\alpha} = \left(\underline{x}^{\alpha}(t_0), \overline{x}^{\alpha}(t_0)\right) \\ [\tilde{x}(t_f)]^{\alpha} = \left(\dot{\underline{x}}^{\alpha}(t_f), \overline{x}^{\alpha}(t_f)\right) \end{cases}$$

Therefore, the approximate solution of fuzzy nonlinear optimal control problem of (39) can be calculated as optimal solution of the linear programming problem of (53).

IV. ILLUSTRATIVE EXAMPLE

Example 4.1. Consider fuzzy nonlinear optimal control problem of (54).

$$\mathrm{Min}\ \int_{0}^{1} |\sin(t)\, u(t)|\, dt$$

$$\text{s.t:}\begin{cases} \dot{\tilde{x}}(t) = \cos(t)\, \tilde{x}(t) + u^2(t) \\ \tilde{x}(0) = \tilde{1} \\ \tilde{x}(1) = \tilde{2} \\ 0 \le u \le 1 \end{cases} \qquad (54)$$

where, $\tilde{1} = (1,1,1)$ and $\tilde{2} = (2,1,1)$. According to definition 2.6.

$\tilde{1} = (\alpha, 2 - \alpha)$ and $\tilde{2} = (\alpha + 1, 3 - \alpha)$.

In this example, n is set to 100. The fuzzy state variable and control variable have been shown in figures 2 and 3.

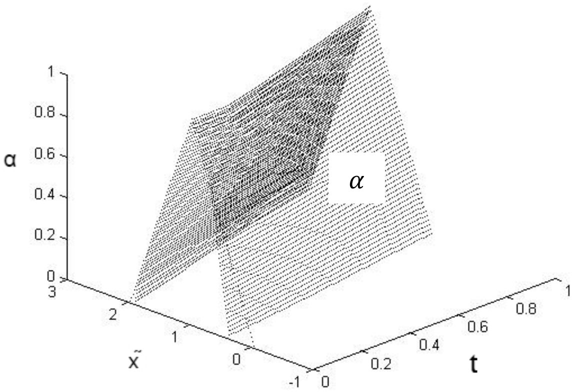

Fig. 2. The fuzzy state variable $\tilde{x}(t)$ of example 4.1

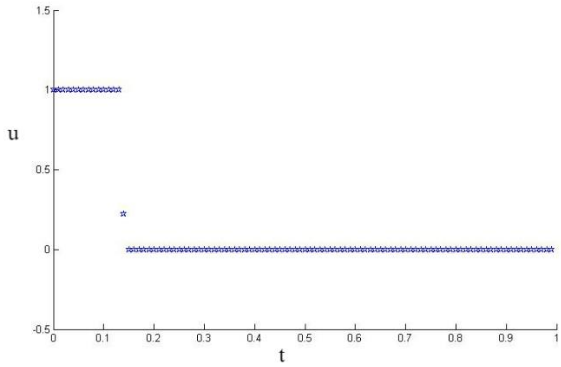

Fig. 3. The control function $\mathbf{u(t)}$ of example 4.1

Example 4.2. Consider fuzzy nonlinear optimal control problem of (55).

$$\mathrm{Min}\ \int_{0}^{1} \|\sin(t)\, \tilde{x}(t)\|_1 dt$$

$$\text{s.t:}\begin{cases} \dot{\tilde{x}}(t) = \cos(2\pi t)\, \tilde{x}(t) - \tan\left(\frac{\pi}{8}u^3(t) + t\right) \\ \tilde{x}(0) = \tilde{1} \\ \tilde{x}(1) = \tilde{0} \\ 0 \le u \le 1 \end{cases} \qquad (55)$$

In this example, n is set to 500. The fuzzy state variable and control variable have been shown in figures 4, 5 and 6.

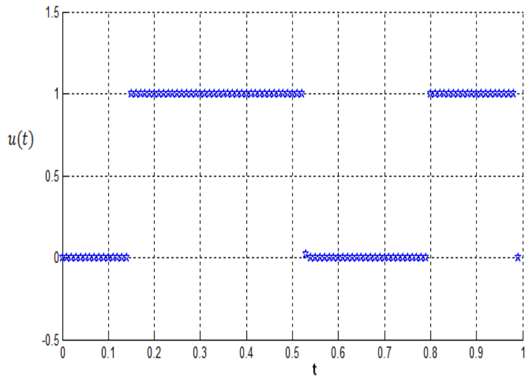

Fig. 4. The control function $\mathbf{u}(\mathbf{t})$ of example 4.2

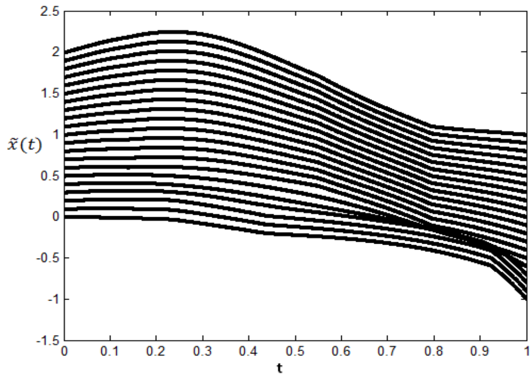

Fig. 5. The fuzzy state variable $\tilde{x}(t)$ of example 4.2

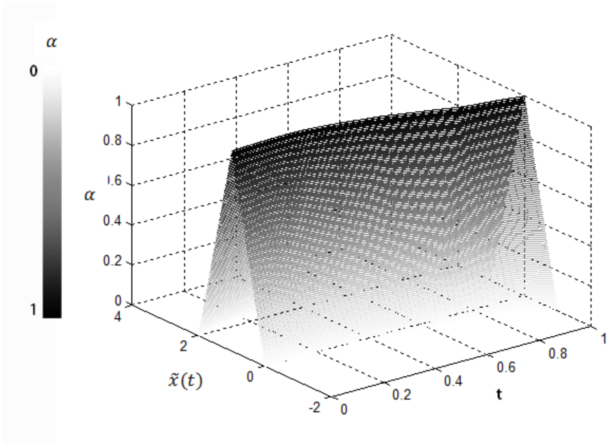

Fig. 6. The fuzzy state variable $\tilde{x}(t)$ of example 4.2

V. Conclusion

In this paper, a class of fuzzy nonlinear optimal control problems is studied and a new approach is proposed based on $\alpha -$cut sets and using a novel piecewise linearization approach to reduce this typical fuzzy nonlinear optimal control problem to an approximated linear programing problem. The optimal solution of the later linear programing problem is the best approximated solution of the original fuzzy nonlinear optimal control problem. The class that has been considered in this paper contains crisp control variable and the presented approach has to be improved to the class of fuzzy nonlinear optimal control problems that contains fuzzy control parameter.

VI. References

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